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ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS CORRESPONDING TO POLYNOMIAL SZEGŐ MEASURE WITH AN INFINITE DISCRETE PART

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Abstract. The asymptotics behavior orthogonal polynomials have been in the spotlight since the result of G. Szegő in 1921. In this paper we study the pointwise asymptotics inside the unit disk for orthogonal polynomials with respect to a polynomial Szegő measure with an infinite masses points.

1. Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and let p be a trigonometric polynomial, non-negative on $\mathbb{T}$. We say that a measure μ on $\mathbb{T}$ belongs to the polynomial Szegő class (denoted by $\mu \in$ (pS)) if $d\mu = \mu'_{ac}dm + d\mu_s$, where μ_{ac} is the absolutely continuous part of μ and $d\mu_s$ is singular and,

$$\int_{\mathbb{T}} p(t) \log \mu'_{ac}(t) dm(t) > -\infty.$$

Where m is the probability Lebesgue measure on $\mathbb{T}$ i.e. $dm(t) = dt/(2\pi i t) = 1/(2\pi)d\theta$, $t = e^{i\theta} \in \mathbb{T}$. Denote by P_n the set of polynomials of degree at most n and $\varphi_n(z) = k_n z^n + ... \in P_n$ $(k_n > 0)$ be the polynomial of degree n orthonormal with respect to μ i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z)\overline{z^k} d\mu(\theta) = k_n^{-1}\delta_{kn}. \tag{1.1}$$

$k = 0, 1, ..., n$, $z = e^{i\theta}$, where δ_{kn} is the Kronecker's symbol.
For the system $\{\varphi_n\}_{n\in\mathbb{N}}$, one gets

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$$\frac{1}{2\pi}\int_0^{2\pi}\varphi_n(z)\overline{\varphi_m(z)}d\mu(\theta)=\delta_{mn},\quad m,n,...,\ z=e^{i\theta}. \tag{1.2}$$

In 2006 Denisov and Kupin in [3] have obtained pointwise asymptotics in the open unit disk $\mathbb{D}$ for the associated orthonormal polynomials $\varphi_n(z)$ and proved these asymptotics in L2-sense on the unit circle . For the polynomial Szego class measure they have introduced the functions :

$$\widetilde{D}(z)=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}K(e^{i\theta},z)\log\mu'_{ac}(e^{i\theta})d\theta\right\}, \tag{1.3}$$

$$\widetilde{\varphi}_n^*(z)=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}K(e^{i\theta},z)\log\left|\varphi_n^*(e^{i\theta})\right|d\theta\right\}, \tag{1.4}$$

$$\widetilde{\psi}_n^*(z)=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}K(e^{i\theta},z)\log\left|\psi_n^*(e^{i\theta})\right|d\theta\right\}, \tag{1.5}$$

where $K(e^{i\theta},z)$ is the modified Schwarz kernel defined by

$$K(t,z)=\frac{t+z}{t-z}\frac{q(t)}{q(z)}=\frac{t+z}{t-z}\frac{q_0(t)}{q_0(z)},$$

with $q(t)=\prod_{k=1}^N(t-\varsigma_k)^{2K_k}/t^{N'}$, $N'=\sum_k K_k$, $t=e^{i\theta}$ and $q(t)=Cq_0(t)$. The constant C equals $(\prod_k(-\varsigma_k)^{K_k})^{-1}$, so that $|C|=1$ and $q(t)=\prod^k|t-\varsigma_k|^{2K_k}=p(t)$ for $t\in T$. The functions $\{\widetilde{\varphi}_n^*\}$ are called the modified reversed orthogonal polynomials with respect to μ.

Theorem 1.1. [3] *Let $\mu\in(pS)$. Then*

$$\lim_{n\to\infty}\widetilde{D}(z)\widetilde{\varphi}_n^*(z)=1.$$

for every $z\in\mathbb{D}$.

The proof of this theorem is largely inspired by the classical proof of (Theorem [13]).
In 2011, Khaldi and Guezane-Lakoud in [7] have been generalized this theorem in the case where the measure $v_l\in(pS)$ on $\mathbb{T}\cup\{z_k\}_{k=1}^l$ perturbed by a finite Blaschke sequence of point masses outside the unit circle, where the masses $A_k>0$ for $k=1,...,l$; and $\delta(z-z_k)$ is the Dirac measure supported at the point z_k.
We denote by $\{\psi_k\}$ the system of orthonormal polynomials associated

to v_l .
$\{\psi_k\}_{n\in\mathbb{N}}$ satisfy the following orthonormality relations:

$$\psi_n(z) = \gamma_n^l z^n + ...(\gamma_n^l > 0),$$

$$\frac{1}{2\pi}\int_0^{2\pi} \psi_n(z)\overline{\psi_m(z)}d\mu(\theta) + \sum_{k=1}^{l} A_k\psi_n(z_k)\overline{\psi_m(z_k)} = \delta_{mn}. \tag{1.6}$$

$$m, n, ..., z = e^{i\theta}.$$

Khaldi and Guezane-Lakoud have invested the ratio for the two orthonormal polynomials $\{\psi_k\}$ and $\{\varphi_k\}$ to give,

THEOREM 1.2. [6] *Let $v_l = \mu + \sum_{k=1}^{l} A_k\delta(z - z_k)$ such that $\mu \in (pS)$. Associate with the measure v_l the functions $\tilde{D}(z)$ and $\tilde{\psi}_n^*(z)$ given by (1.3) , (1.5) then we have*

$$\lim_{n\to\infty} \widetilde{D}(z)\widetilde{\psi}_n^*(z) = 1,$$

for every $z \in \mathbb{D}$.

The asymptotic behavior of the polynomials $\{\psi_k\}$ has been established by Li and Pan [13] in the case where the measure μ is not absolutely continuous.
Consider now the measure v on $\mathbb{T} \cup \{z_k\}_{k=1}^{\infty}$, z_k are fixed points outside $\mathbb{D}$

$$v = \mu + \sum_{k=1}^{\infty} A_k\delta(z - z_k),$$

where the masses A_k satisfy

$$A_k > 0,\ \sum_{k=1}^{\infty} A_k < \infty \text{ and } \sum_{k=1}^{\infty}(|z_k| - 1) < +\infty, \tag{1.7}$$

for $k = 1, ...$ and $\delta(z-z_k)$ is the Dirac measure supported at the point z_k.

Note that the system $\{\Phi_k\}_{n\in\mathbb{N}}$ satisfy the following orthonormality relations:

$$\Phi_n(z) = \gamma_n z^n + ...(\gamma_n > 0),$$

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi_n(z)\overline{\Phi_m(z)}d\mu(\theta) + \sum_{k=1}^{\infty} A_k\Phi_n(z_k)\overline{\Phi_m(z_k)} = \delta_{mn}. \tag{1.8}$$

$$m, n, ..., z = e^{i\theta}.$$

The purpose of this paper is to study the pointwise asymptotics inside the unit disk for orthogonal polynomials with respect to a measure from polynomial Szegő class and perturbed by an infinite masses points outside the unit circle. A similar study has been done by Khaldi and R. Benzine in ([6]) where μ is a positive measure on the unit circle satisfying the Szegő condition with an infinite discrete part. To get the asymptotic formula of $\Phi_n(z)$ they have proved two intermediate results.

2. Preliminaries

We say that μ is a Szegő measure (notation: $\mu \in$ (S)) if its singular part μ_s is arbitrary and

$$\int_{\mathbb{T}} \log \mu'_{ac} dm > -\infty$$

where μ'_{ac} is the density of the absolutely continuous part of μ and $dm = dm(t) = d\theta/(2\pi),\ t = e^{i\theta} \in \mathbb{T}$ is the normalized Lebesgue measure on $\mathbb{T}$.
Now we define Szegő polynomial class (Sp) as follow: Let p be a trigonometric polynomial such that $p(t) \geq 0,\ t \in \mathbb{T}$. Without loss of generality we can assume that

$$p(t) = \prod_{N}^{k=1} (t - \xi_k)^{2m_k}$$

where ξ_k are points on $\mathbb{T}$ and $m_k > 0$ are their multiplicities. We say that a measure μ belongs to the polynomial Szegő class (denoted by $\mu \in (pS)$) if

$$\int_0^{2\pi} p(e^{i\theta}) \log \mu'_{ac}(e^{i\theta}) d\theta > -\infty,$$

We say that μ belongs to the Erdős class ($\mu \in (E)$) if $\mu'_{ac} > 0$ a.e. on $\mathbb{T}$.
The following relations are true (see [11]).

$$S \subset pS \subset E$$

.

2.1. The space $H^2(G, \widetilde{p})$

In what follows, we suppose that the absolutely continuous part of μ with respect to the Lebesgue measure m satisfies the generalized Szegő condition:

LEMMA 2.1. *Let $\mu \in (pS)$ and the function $\widetilde{D}$ be defined in (3), then*

1. $\widetilde{D}(z) \in H^2(D)$,
2. $\widetilde{D}(z) \neq 0$ *for* $|z| < 1$,
3. $\left|\widetilde{D}(t)\right|^2 = \mu'_{ac}(t)$ *a.e. on* T,
4. $\widetilde{D}(0) > 0$.

Proof. Consider the Poisson integral associated with the $\frac{q(e^{i\theta})}{q(z)} \log \mu'_{ac}(e^{i\theta})$ function which we denote by:

$$u(r,x) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{1-r^2}{1-2r\cos(x-\theta)+r^2}\frac{q(e^{i\theta})}{q(z)}\log \mu'_{ac}(e^{i\theta})\right\}d\theta$$

The function u is harmonic in the unit disk D since $q(t) = \prod^k |t-\varsigma_k|^{2K_k} = p(t)$ and $p(e^{i\theta})\log \mu'_{ac}(e^{i\theta})d\theta \in L^1([0,2\pi], d\theta)$.
Now consider the holomorphic function $h(z)$ of which $u(r,x)$ is the real part. and require that $h(0)$ be real to have the uniqueness of h. The function sought will therefore be $g(z) = \exp h(z)$. It is clear that $Re\widetilde{D}(z) = Reg(z)$; $(z = re^{ix}, r \in [0,1[)$, so that $\widetilde{D}(z) = g(z)$.

$$\begin{aligned}\left|\widetilde{D}(z)\right| &= \left|\exp\{Reh(z) + Imh(z)\}\right|,\\ &= \exp\{Reh(z)\},\\ &= \exp\{u(r,x)\}.\end{aligned}$$

Then, for $(z = re^{ix}, r \in [0,1[)$, we have

$$\begin{aligned}\left|\widetilde{D}(re^{ix})\right|^2 &= 2\exp\{u(r,x)\},\\ &\leq \frac{1}{\pi}\int_0^{2\pi}\left\{\frac{q(e^{i\theta})}{q(z)}\log \mu'_{ac}(e^{i\theta})\frac{1-r^2}{1-2r\cos(x-\theta)+r^2}\right\}d\theta.\end{aligned}$$

By integrating with respect to x we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{D}(re^{ix})\right|^2 dx &\leq \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{\pi}\int_0^{2\pi}\frac{q(e^{i\theta})}{q(z)}\log \mu'_{ac}(e^{i\theta})A(x,\theta)d\theta\right)dx,\\ &= \frac{1}{\pi}\int_0^{2\pi}\frac{q(e^{i\theta})}{q(z)}\mu'_{ac}(e^{i\theta})\left(\frac{1}{2\pi}\int_0^{2\pi}A(x,\theta)dx\right)d\theta,\\ &= \frac{1}{\pi}\int_0^{2\pi}\frac{q(e^{i\theta})}{q(z)}\log \mu'_{ac}(e^{i\theta})d\theta = C,\end{aligned}$$

with $A(x,\theta) = \frac{1-r^2}{1-2r\cos(x-\theta)+r^2}$, which proves the first point . The second point is obvious. The third point is already proved in [11] . To prove the last point we note that $\widetilde{D}(0) = \exp\{h(0)\} > 0$ ($h(0) \in \mathbb{R}$ by construction). $\square$

Let G denotes the following set of complex numbers

$$G = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}.$$

Definition 2.2. We say that $f \in H^2(G)$ if f is analytic in G and $\int_{C_r} |f(z)|\,|dz| \leq C,\ r > 1,\ C_r = \{z \in \mathbb{C} : |z| = r\}$, and C is a constant independent of r.

Lemma 2.3. *Let $\mu \in (pS)$ and the function $\widetilde{D}^{out}$ define outside the unit circle by*

$$\widetilde{D}^{out}(w) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{w + e^{-i\theta}}{w - e^{-i\theta}}\frac{q(e^{i\theta})}{q(w)} \log \mu'_{ac}(e^{i\theta})d\theta\right\}.$$

Then

1. $\widetilde{D}^{out}(w) \in H^2(G)$,
2. $\widetilde{D}^{out}(w) \neq 0$ *for* $w \in G$,
3. $\left|\widetilde{D}^{out}(t)\right|^2 = \mu'_{ac}(t)$ *a.e. on* T,
4. $\widetilde{D}^{out}(\infty) > 0$.

Proof. Consider the function $\widetilde{D}$ defined in (3), and construct the function $\widetilde{D}^{out}$ as follows:

$$\begin{cases} \widetilde{D}^{out}(w) = \widetilde{D}(\frac{1}{w}) \ for\ w \in G/\{\infty\}. \\ \widetilde{D}^{out}(\infty) = \widetilde{D}(0). \end{cases}$$

Then, the proof immediately follows from the above Lemma.

$\square$

Definition 2.4. We say that $f \in H^2(G, \widetilde{p})$ if $f(z)$ is analytic in G and $(f.D) \in H^2(G)$.

The space $L^2(\mathbb{T}, \mu'_{ac}\,|d\xi|)$ is the space of functions f defined on the unit circle $\mathbb{T}$, with values in $\mathbb{C}$ and for which $\int_{-\pi}^{+\pi} \left|f(e^{i\theta})\right|^2 \mu'_{ac}(\theta)d\theta < +\infty$. Let f and g be in $L^2(\mathbb{T}, \mu'_{ac}\,|d\xi|)$, we define

$$\langle f, g\rangle_{L^2(\mathbb{T},\mu'_{ac}|d\xi|)} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\mu'_{ac}(\theta)d\theta,$$

$$\|f\|^2_{L^2(\mathbb{T},\mu'_{ac}|d\xi|)} = \langle f, f\rangle_{L^2(\mathbb{T},\mu'_{ac}|d\xi|)},$$

then $(L^2(\mathbb{T}, \mu'_{ac}\,|d\xi|), \|.\|_{L^2(\mathbb{T},\mu'_{ac}|d\xi|)})$, is a Hilbert space.

The properties of the space $H^2(G, \mu'_{ac})$ are given in the following theorem:

THEOREM 2.5. *Let $f \in H^2(G, \widetilde{p})$. Then f has a.e. in $\mathbb{T}$ an angular limit which is denoted by $\widetilde{f}$, $\widetilde{f}(e^{i\theta}) = \lim_{z \to e^{i\theta}} f(z)$. Moreover,*

1. $\tilde{f} \in (L^2(\mathbb{T}, \mu'_{ac}\,|d\xi|)$
2. $(H^2(G, \widetilde{p}), \|.\|_{\mu'_{ac}})$ *is a Hilbert space, where*
$$\|f\|^2_{\mu'_{ac}} = \langle f, f\rangle_{\mu'_{ac}},$$
$$\langle f, g\rangle_{\mu'_{ac}} = \left\langle \widetilde{f}, \widetilde{g} \right\rangle_{L^2(\mathbb{T},\mu'_{ac}|d\xi|)} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \widetilde{f}(e^{i\theta})\overline{\widetilde{g}(e^{i\theta})}\mu'_{ac}(\theta)d\theta.$$
For any f, g in $\in H^2(G, \widetilde{p})$.
3. *if $f \in H^2(G, \widetilde{p})$, then for every compact set $K \subset G$, there is a constant $C(K)$ ($C(K)$ depends only on K) such that*
$$\sup_{z\in K} |f(z)| \leq C(K)\, \|f\|_{\mu'_{ac}}$$

For the proof see [6].

2.2. Extremal properties of the orthogonal polynomials

We denote by Q_n the monic polynomials of degree exactly equal to n. Define $m_n(\mu)$, $m_n(v_l)$, $m_n(v)$, μ and $\hat{\mu}$ as the extremal values of the following problems:
(2.1)
$$m_n(\mu) = \left(\frac{1}{k_n}\right)^2 = \left\|\frac{1}{k_n}\varphi\right\|^2_\mu = min\left\{\|Q_n\|^2_\mu\,; \varphi_n = z^n + ..., Q_n(\infty) = 1\right\},$$
where
$$\|Q_n\|^2_\mu = \frac{1}{2\pi}\int_0^{2\pi} \left|Q_n(e^{i\theta})\right|^2 d\mu(\theta).$$

(2.2)
$$m_n(v_l) = \left(\frac{1}{\gamma^l_n}\right)^2 = \left\|\frac{1}{\gamma^l_n}\psi_n\right\|^2_{v_l} = min\left\{\|Q_n\|^2_{v_l}\,; Q_n = z^n + ..., Q_n(\infty) = 1\right\},$$
where
$$\|Q_n\|^2_{v_l} = \frac{1}{2\pi}\int_0^{2\pi} \left|Q_n(e^{i\theta})\right|^2 d\mu(\theta) + \sum_{k=1}^{l} A_k\,|Q_n(z_k)|^2,$$

(2.3)

$$m_n(v) = \left(\frac{1}{\gamma_n}\right)^2 = \left\|\frac{1}{\gamma_n}\Phi_n\right\|_v^2 = min\left\{\|Q_n\|_v^2 ; Q_n = z^n + ..., Q_n(\infty) = 1\right\}$$

where

$$\|Q_n\|_v^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|Q_n(e^{i\theta})\right|^2 d\mu(\theta) + \sum_{k=1}^{\infty} A_k \left|Q_n(z_k)\right|^2 .$$

(2.4) $$\mu(\widetilde{p}) = inf\left\{\|\varphi_n\|^2_{H^2(G,\widetilde{p})} : \varphi_n \in H^2(G,\widetilde{p}); \varphi_n(\infty) = 1\right\}, and$$

(2.5)

$$\hat{\mu}(v) = inf\left\{\|\varphi_n\|^2_{H^2(G,\widetilde{p})} : \varphi_n \in H^2(G,\widetilde{p}); \varphi_n(\infty) = 1; \varphi_n(z_k) = 0, k = 1, 2, ...\right\}.$$

Denote by $\widehat{\varphi}$ and φ^{∞} the optimal solutions of the extremal problems (2.4) and (2.5) , respectively.
We are now ready to state the lemmas which we need in the sequel.

LEMMA 2.6. [1] *Let $\varphi \in H^2(G,\widetilde{p})$ such that $\varphi(\infty) = 1$ and $\varphi(z_k) = 0$, $k = 1, 2, ...$, and let*

$$B_{\infty}(z_k) = \prod_{k=1}^{\infty} \frac{z - z_k}{z\overline{z_k} - 1}\frac{|z_k|^2}{z_k}$$

be the Blashke product, then

$$B_{\infty} \in H^2(G,\widetilde{p}),\ B_{\infty}(\infty) = 1,\ \left|\widetilde{B}_{\infty}(e^{i\theta})\right| = \prod_{k=1}^{\infty} |z_k|$$

with

$$\widetilde{B}_{\infty}(e^{i\theta}) = \lim_{z \to e^{i\theta}} B_{\infty}(z),\ \frac{\varphi}{B_{\infty}} \in H^2(G,\widetilde{p})$$

LEMMA 2.7. [1] *The extremal functions φ^* and $\widehat{\varphi}^*$ are connected by the relations*

$$\varphi^{\infty}(z) = B_{\infty}(z).\widehat{\varphi}(z) \;\; and \;\; \widehat{\mu}(v) = \left[\prod_{k=1}^{\infty} |z_k|\right]^2 \mu(\widetilde{p})$$

3. Main results

These are the main results of this paper.

THEOREM 3.1. *let* $v = \mu + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ *such that* μ *is polynomial Szegő on the unit circle,*

$$\sum_{k=1}^{\infty} A_k < +\infty, \quad A_k > 0,$$

$$\sum_{k=1}^{\infty} (|z_k| - 1) < +\infty, \quad |z_k| > 1.$$

Then

$$\lim_{l \to \infty} m_n(v_l) = m_n(v), \tag{3.1}$$

$$\lim_{l \to \infty} \gamma_n^l = \gamma_n. \tag{3.2}$$

Proof. The extremal property of $\frac{1}{\gamma_n^l}\psi_n$ and the fact that $A_k > 0$ implies that

$$m_n(v_l) \le \frac{1}{2\pi} \int_0^{2\pi} \left|\Phi_n(e^{i\theta})\right|^2 d\mu(\theta) + \sum_{k=1}^{l} A_k \left|\Phi_n(z_k)\right|^2 \le m_n(v), and$$

thus

$$m_n(v_l) \le m_n(v). \tag{3.3}$$

On the other hand, the extremal property of $\frac{1}{\gamma_n}\Phi_n$ implies that

$$m_n(v) \le \frac{1}{2\pi} \int_0^{2\pi} \left|\psi_n(e^{i\theta})\right|^2 d\mu(\theta) + \sum_{k=1}^{\infty} A_k \left|\psi_n(z_k)\right|^2 \tag{3.4}$$

$$= m_n(v_l) + \sum_{k=l+1}^{\infty} A_k \left|\psi_n(z_k)\right|^2. \tag{3.5}$$

According to the reproducing property of the kernel function $K_n(\xi, z)$ (see [13]), and $\psi_n(z) \in P_n$, we have

$$\psi_n(z_k) = \frac{1}{2\pi} \int_0^{2\pi} \psi_n(e^{i\theta}) \overline{K_n(\xi, z_k)} d\mu(\theta) \tag{3.6}$$

The Scharwz inequality implies that

$$\psi_n(z_k) = \frac{1}{2\pi}\int_0^{2\pi} \psi_n(e^{i\theta})\overline{K_n(\xi, z_k)}d\mu(\theta) \tag{3.7}$$

$$\leq m_n(v_l).\frac{1}{2\pi}\int_0^{2\pi} |K_n(\xi, z_k)|^2 \, d\mu(\theta) \tag{3.8}$$

and the fact that $K_n(\xi, z_k) \in P_n$

$$|\psi_n(z_k)|^2 \leq m_n(v_l).K_n(z_k, z_k) \tag{3.9}$$

The inequalities (1.7), (3.5), and (3.9) imply

$$m_n(v) \leq m_n(v_l) + \sum_{k=l+1}^{\infty} A_k m_n(v_l).K_n(z_k, z_k) \tag{3.10}$$

$$= m_n(v_l)\left[1 + \sup_{k\geq l+1} K_n(z_k, z_k) \sum_{k=l+1}^{\infty} A_k\right]. \tag{3.11}$$

so we get

$$\frac{m_n(v)}{m_n(v_l)} \leq 1 + \delta_l \text{ where } \delta_l \to 0, \ l \to \infty. \tag{3.12}$$

Using (3.3) and (3.12), we obtain (3.1).Thus, we obtain (3.2) and the theorem is proved. □

THEOREM 3.2. *let $v = \mu + \sum_{k=1}^{\infty} A_k\delta(z - z_k)$ such that μ is polynomial Szegő measure on the unit circle and;*

1.

$$\sum_{k=1}^{\infty} A_k < +\infty, \quad A_k > 0, \tag{3.13}$$

2.

$$\sum_{k=1}^{\infty} (|z_k| - 1) < +\infty, \quad |z_k| > 1, \tag{3.14}$$

3.

$$\frac{m_n(v_l)}{m_n(\mu)} \leq \left(\prod_{k=1}^{l} |z_k|^2\right) \tag{3.15}$$

Then

$$\lim_{n\to\infty} m_n(v) = \widehat{\mu}(v). \tag{3.16}$$

Proof. By passing to the limit when l tends to infinity and using theorem (3.1) and the inequality (3.15), we obtain

$$m_n(v) \leq \left(\prod_{k=1}^{\infty} |z_k|\right)^2 m_n(\mu).$$

This implies, by lemma(2.7) , that

$$\limsup_{n\to\infty} m_n(v) \leq \left(\prod_{k=1}^{\infty} |z_k|\right)^2 \mu(\widetilde{p}) = \widehat{\mu}(v).$$

On the other hand, we can find function L in $H^2(G,\widetilde{p})$, (see [6],[10]), were

$$\widehat{\mu}(v) \leq \|L(z)\|^2_{H^2(G,\widetilde{p})} \leq \liminf_{n\to\infty} m_n(v),$$

$$\widehat{\mu}(v) \leq \liminf_{n\to\infty} m_n(v).$$

The theorem is then proved. □

THEOREM 3.3. *Let $v = \mu + \sum_{k=1}^{\infty} A_k\delta(z - z_k)$ with μ as in the preceding theorem and such that*

$$\sum_{k=1}^{\infty} A_k < +\infty,\ A_k > 0,\ \sum_{k=1}^{\infty}(|z_k| - 1) < +\infty \ for\ |z_k| > 1,$$

$$\frac{k_n}{\gamma_n^l} \leq \prod_{k=1}^{l} |z_k|\,.$$

Let $\{\Phi_n\}_{n=1}^{\infty}$ be the system of orthonormal polynomials associated to v satisfying relations (1.8). Then

$$\lim_{n\to\infty} \left\| \frac{1}{\gamma_n}\Phi_n - \varphi^{\infty} \right\|^2_{H^2(G,\widetilde{p})} = 0 \tag{3.17}$$

Proof. Put

$$H_n = \frac{1}{2}\left(\frac{1}{\gamma_n}\Phi_n + \varphi^{\infty}\right).$$

Then

$$\lim_{n\to\infty} H_n(\infty) = 1,\ H_n(z_k) = 0,\ k = 1, ...$$

This implies that

$$\liminf_{n\to\infty} \|H_n\|^2_{H^2(G,\widetilde{p})} \leq \hat{\mu}(v)$$

Using the parallelogram law in $H^2(G,\widetilde{p})$, we obtain

$$\left\|\frac{1}{\gamma_n}\Phi_n-\varphi^\infty\right\|^2_{H^2(G,\widetilde{p})}=2\left(\left\|\frac{1}{\gamma_n}\Phi_n\right\|^2_{H^2(G,\widetilde{p})}+\|\varphi^\infty\|^2_{H^2(G,\widetilde{p})}\right)-4\|H_n\|^2_{H^2(G,\widetilde{p})}$$

Hence

$$\limsup_{n\to\infty}\|H_n\|^2_{H^2(G,\widetilde{p})}\leq 2\left(\hat{\mu}(v)+\hat{\mu}(v)\right)-4\hat{\mu}(v)=0$$

so that

$$\lim_{n\to\infty}\left\|\frac{1}{\gamma_n}\Phi_n-\varphi^\infty\right\|^2_{H^2(G,\widetilde{p})}=0$$

□

References

[1] R. Benzine, *Asymptotic behavior of orthogonal polynomials corresponding to a measure with infinite discrete part off a curve* , Approx. Theory,. **89** (1997), no. 2, 257-265.

[2] A. Cachafeiro and F. Marcellan, *Perturbation in Toeplitz matrices,* J.Math.Anal.Appl., (1991).

[3] S. Denisov and S. Kupin, *Asymptotics of the orthogonal polynomials for the Szego class with a polynomial weight,* J. Approx. Theory., **139** (2006), 8-28.

[4] V. Kaliaguine and R. Benzine, *Sur la formule asymptotique des polynomes orthogonaux associes a une mesure concentree sur un contour plus une partie discrete finie* , Bull. Soc. Math. Belg., **41** (1989), 29-46.

[5] R. Khaldi and R. Benzine, *Asymptotic behavior of orthogonal polynomials corresponding to a measure with infinite discrete part off an arc,* Int. J. Math. Math. Sci., **26** (2001), no. 8, 449-455.

[6] R. Khaldi and R. Benzine, *Asymptotics for orthogonal polynomials off the circle,* J. appl. math. phys., **2004** (2004), no. 1, 37-53.

[7] R. Khaldi and A. Guezane-Lakoud, *Asymptotics of Orthogonal Polynomials With a Generalized Szego Condition,* int. J. Open Problems Complex Analysis., **3** (2011) no. 1, ISSN 2074-2827.

[8] X. Li and K. Pan, *Asymptotic behavior of orthogonal polynomials corresponding to measure with discrete part off the unit circle,* J. Aprox. Theory., **79** (1994), 54-71.

[9] E. A. Rahmanov, *On the asymptotics of the ratio of orthogonal polynomials,* II Math. USSD-SB., **46** (1983), 105-117.

[10] W. Rudin, *Real and Complex Analysis,* McGraw-Hill, New York,(1966).

[11] B. Simon, *Orthogonal polynomials on the unit circle. Part 1,* American Mathematical Society, Providence, (2005).

[12] G. Szego , *Uber die Enttwickelung einer analytischen Funktion nach den polynomen einers orthogonal systems,* Mathematische zeitchrift, **82** (1921), 188-212.

[13] G.Szego, *Orthogonal Polynomials,* American Mathematical Society, Providence, (1975).

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